EXERCISES [MAI 2.15-2.16]

POWER AND EXPONENTIAL MODELS

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a)

$y = 0.2x$	y is proportional to x
$y = 3.7x^2$	y is proportional to the square of x
$y = 2.15/x$	y is inversely proportional to x
$y = 1/(3x^2)$	y is inversely proportional to the square of x

(b)

$P = kQ$	P is proportional to Q
$P = kQ^3$	P is proportional to the cube of Q
$P^2 = kQ^3$	The square of P is proportional to the cube of Q
$PQ^2 = k$	P is inversely proportional to the square of Q

2. (a) The distance travelled is proportional to the square of the time.

(b) If g remains constant: the force is **proportional** to the mass.

If m remains constant: the force is **proportional** to the acceleration.

If F remains constant: the mass is **inversely proportional** to the acceleration.

3. $S = \dfrac{k}{T^2}$

$S = 10$ when $T = 0.2$: $\quad 10 = \dfrac{k}{0.2^2} \Rightarrow k = 0.4$.

Hence, $S = \dfrac{0.4}{T^2}$

4. (a) $P = ax^n$: $\quad 10 = a \times 1^n \Rightarrow a = 10$,

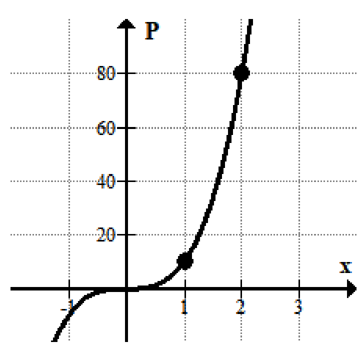
$80 = a \times 2^n \Rightarrow 80 = 10 \times 2^n \Rightarrow 2^n = 8 \Rightarrow n = 3$

Hence $P = 10x^3$

(b) $P = 10 \times 1.5^3 \Rightarrow P = 33.75$

(c) $20 = 10 \times x^3 \Rightarrow x^3 = 2 \Rightarrow x = \sqrt[3]{2}$

(d)



5. (a) $P = aQ^n$: $1.2 = a \times 2^n$ (1)

$4.8 = a \times 4^n$ (2)

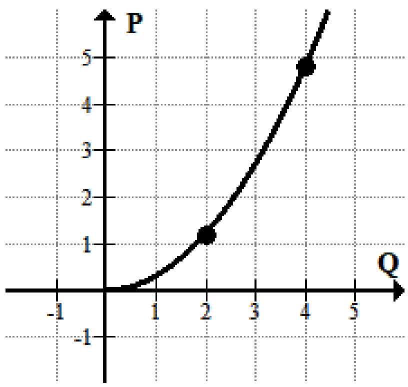
Divide (2) by (1): $\dfrac{4^n}{2^n} = \dfrac{4.8}{1.2} \Rightarrow 2^n = 4 \Rightarrow n = 2$

(1) gives $1.2 = a \times 2^2 \Rightarrow a = 0.3$

Hence, $P = 0.3Q^2$

(b) $2.7 = 0.3Q^2 \Rightarrow Q = 3$

(c)



6. (a) $P = \dfrac{a}{Q^n}$

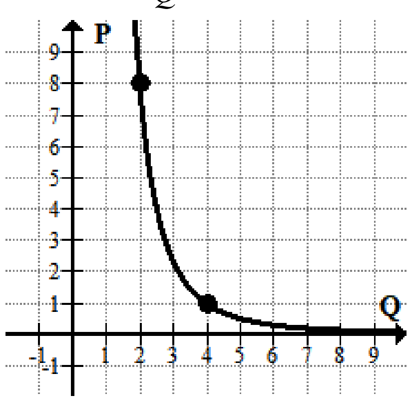
$8 = \dfrac{a}{2^n} \Rightarrow a = 8 \times 2^n$ (1)

$1 = \dfrac{a}{4^n} \Rightarrow a = 4^n$ (2)

(1) and (2) give: $4^n = 8 \times 2^n \Rightarrow 2^n = 8 \Rightarrow n = 3$

(1) gives $a = 8 \times 2^3 \Rightarrow a = 64$

(b) Hence, $P = \dfrac{64}{Q^3}$



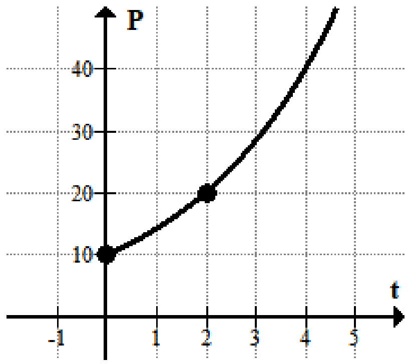
(c) (i) $0.001 = \dfrac{64}{Q^3} \Rightarrow Q = 40$ (or by GDC graph)

(ii) $Q = 41$

7. (a) $A = 10$

(b) $20 = 10e^{2k} \Leftrightarrow e^{2k} = 2 \Leftrightarrow 2k = \ln 2 \Leftrightarrow k = \dfrac{\ln 2}{2} = 0.347$

(c) $P = 10e^{0.347t}$



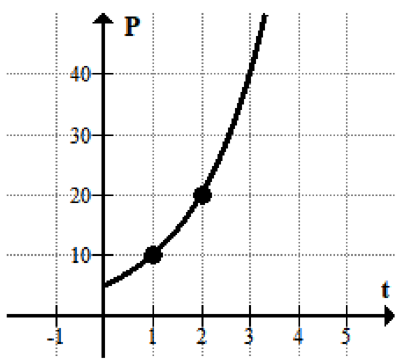
(d) GDC-regression gives the same result $P = 10e^{0.347t}$

8. (a) $P_0 e^k = 10$ and $P_0 e^{2k} = 20$

Divide the two relations: $\dfrac{P_0 e^{2k}}{P_0 e^k} = \dfrac{20}{10} \Leftrightarrow e^k = 2 \Leftrightarrow k = \ln 2 \quad (= 0.693)$

Hence $2P_0 = 10 \Leftrightarrow P_0 = 5$

(b) $P = 5e^{0.693t}$ [OR $P = 5e^{(\ln 2)t} = 5 \times 2^t$]



(c) $30 = 5 \times 2^t \Rightarrow 2^t = 6 \Rightarrow t = \dfrac{\ln 6}{\ln 2} = 2.58496... \cong 2.58$. Hence $t_1 = 2.58$

(d) GDC-regression gives the same result: $P = 5e^{0.693t}$ or $P = 5 \times 2^t$

9. (a) $P_0 e^k + c = 20$, $P_0 e^{2k} + c = 30$, $P_0 e^{3k} + c = 50$

(b) $2^{nd} - 1^{st}$: $P_0 e^{2k} - P_0 e^k = 10$

(c) $3^{rd} - 2^{nd}$: $P_0 e^{3k} - P_0 e^{2k} = 20$

(d) Divide the two relations:

$$\frac{P_0 e^{3k} - P_0 e^{2k}}{P_0 e^{2k} - P_0 e^k} = \frac{20}{10} \Leftrightarrow \frac{P_0 e^{2k}(e^k - 1)}{P_0 e^k (e^k - 1)} = 2 \Leftrightarrow e^k = 2$$. Hence $k = \ln 2$.

Part (b) gives: $P_0 e^{2k} - P_0 e^k = 10 \Rightarrow 4P_0 - 2P_0 = 10 \Rightarrow 2P_0 = 10 \Rightarrow P_0 = 5$

$P_0 e^k + c = 20 \Rightarrow 10 + c = 20 \Rightarrow c = 10$

(e) $P = 5e^{(\ln 2)t} + 10$

10. (a) $P_0e^k + c = 75.50$, $P_0e^{2k} + c = 63.63$, $P_0e^{3k} + c = 53.90$

(b) $2^{nd} - 1^{st}$: $P_0e^{2k} - P_0e^k = -11.87$

$3^{rd} - 2^{nd}$: $P_0e^{3k} - P_0e^{2k} = -9.73$

Divide the two relations:

$$\frac{P_0e^{3k} - P_0e^{2k}}{P_0e^{2k} - P_0e^k} = \frac{-9.73}{-11.87} \Leftrightarrow \frac{P_0e^{2k}(e^k - 1)}{P_0e^k(e^k - 1)} = 0.81971 \Leftrightarrow e^k = 0.81971$$

$k = 0.19880$, so $k = -0.2$ to 1dp

(c) $P_0e^{2k} - P_0e^k = -11.87 \Rightarrow P_0e^{-0.4} - P_0e^{-.02} = -11.87$

$\Rightarrow P_0 = 80$

$P_0e^{-0.2} + c = 75.50 \Rightarrow 80e^{-0.2} + c = 75.50 \Rightarrow 80e^{-0.2} + c = 75.50$

$\Rightarrow c = 10$

(d) $P = 80e^{(-0.2t} + 10$

11. (a) $N = 10$

(b) At $t = 2$, $N = 10e^{0.4(2)}$

$N = 22.3$ (3 sf) Number of leopards = 22

(c) If $N = 100$, then solve $100 = 10e^{0.4t}$

$10 = e^{0.4t}$ hence $\ln 10 = 0.4t$

$t = \dfrac{\ln 10}{0.4} \sim 5.76$ years (3 sf)

(d) $t = \dfrac{\ln 2}{0.4}$ (= 1.73 years)

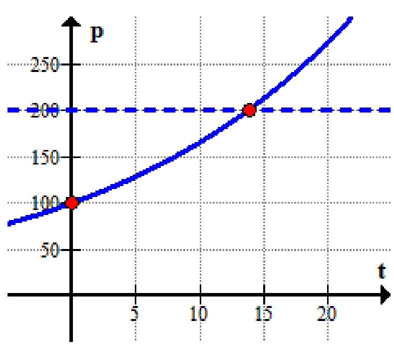
12. (a) $p = 100e^0 = 100$

(b) $200 = 100e^{0.05t}$, so $2 = e^{0.05t}$

$\ln 2 = 0.05t$

$t = \ln 2 / 0.05 = 13.9$

(c)



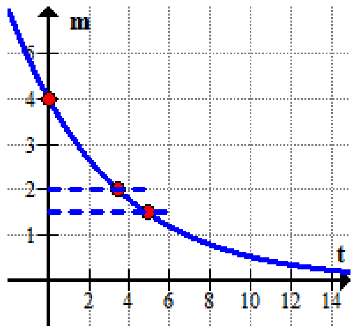
13. (a) Initial mass $\Rightarrow t = 0$, mass = 4

(b) $1.5 = 4e^{-0.2t}$ (or $0.375 = e^{-0.2t}$)
$\ln 0.375 = -0.2t$
$t = 4.90$ hours

(c) When $m = 2$ (half of the initial value)

$t = \dfrac{\ln 0.5}{-0.2}$ (= 3.47 years)

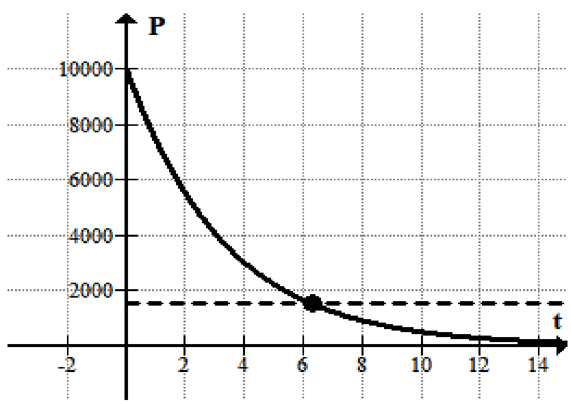
(d)



14. (a) $10\,000e^{-0.3t} = 1500$
Taking logarithms $-0.3t \ln e = \ln 0.15$

$t = \dfrac{\ln 0.15}{-0.3} = 6.32$ 7 (years) (or directly by GDC SolveN or Graph)

(b)



15. (a) $1 = (1/e)e^{5k}$, so $e = e^{5k}$ so $5k=1$ so $k = 0.2$

(b) $100 = \dfrac{1}{e}e^{0.2t}$ $t = \dfrac{\ln 100 + 1}{0.2}$ $(= 28.0)$

(c) See graph on GDC)

16. (a) LINEAR: $P = 37.1Q - 76.6$

POWER: $P = 1.01Q^{2.95}$

QUADRATIC: $P = 10.6Q^2 - 38.2Q + 41.65$

EXPONENTIAL: $P = 1.48e^{0.9Q}$ OR $P = 1.48 \times 2.46^{Q}$

(b) Although the quadratic model is closer to the points (so it could also be a choice) the power model looks better since we know that P is an increasing function of Q.

B. Paper 2 questions (*LONG*)

17. (a) (i) 2420

(ii) $1420 + 100n > 2000 \Rightarrow n > 5.8$

1999 (accept 6^{th} year or $n = 6$)

(b) (i) $1\ 200\ 000(1.025)^{10} = 1\ 536\ 101$ (accept 1 540 000 or 1.54(million))

(ii) $\dfrac{1\ 536\ 101 - 1\ 200\ 000}{1200000} \times 100$ 28.0% (accept 28.3% from 1 540 000)

(iii) $1\ 200\ 000(1.025)^n > 2\ 000\ 000$ (accept an equation)

EITHER *directly by GDC* $n > 20.69$ $n = 21$

OR $n \log 1.025 > \log\left(\dfrac{2}{1.2}\right) \Rightarrow n > 20.69$

2014 (accept 21^{st} year or $n = 21$)

(c) (i) $\dfrac{1200000}{1420} = 845$

(ii) $\dfrac{1200000(1.025)^n}{1420 + 100n} < 600 \Rightarrow n > 14.197$ 15 years

18. (a) $V(5) = 10000 \times (0.933^5) = 7069.8 \ldots = 7070$ (3 sf)

(b) We want t when $V = 5000$

$5000 = 10000 \times (0.933)^t \Rightarrow 9.9949 = t$

After 10 minutes 0 seconds, to nearest second (or 600 seconds).

(c) $0.05 = 0.933^t \Rightarrow t = 43.197$ minutes $\approx 3/4$ hour

(d) (i) $10000 - 10000(0.933)^{0.001} = 0.693$

(ii) Initial flow rate = $\dfrac{dV}{dt} = \dfrac{0.693}{0.001} = 693 = 690$ (2 sf)

OR Later on we may use derivatives to find this rate: $\dfrac{dV}{dt} = 690$ (when $t = 0$)